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THE FOURIER TRANSFORM OF MULTIRADIAL FUNCTIONS

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ABSTRACT. We obtain an exact formula for the Fourier transform of multiradial functions, i.e., functions of the form $\Phi(x) = \phi(|x_1|, \dots, |x_m|)$, $x_i \in \mathbf{R}^{n_i}$, in terms of the Fourier transform of the function ϕ on $\mathbf{R}^{r_1} \times \cdots \times \mathbf{R}^{r_m}$, where r_i is either 1 or 2.

1. Introduction

Let $m \geq 1$, $n_1, \dots, n_m \geq 1$ be integers. Throughout this note, we will adhere to the following notation for the Fourier transform of a function Φ in $L^1(\mathbf{R}^{n_1+\cdots+n_m})$

$$F_{n_1,\dots,n_m}(\Phi)(\xi_1, \dots, \xi_m) = \int_{\mathbf{R}^{n_m}} \cdots \int_{\mathbf{R}^{n_1}} \Phi(x_1, \dots, x_m) e^{-2\pi i(x_1\cdot\xi_1+\cdots+x_m\cdot\xi_m)} dx_1 \cdots dx_m.$$

The function Φ is called multiradial if there exists some function ϕ on $(\mathbf{R}^+ \cup \{0\})^m$ such that

$$\Phi(x_1, \dots, x_m) = \phi(|x_1|, \dots, |x_m|) \tag{1.1}$$

for all $x_i \in \mathbf{R}^{n_i}$, where $|x_j|$ denotes the Euclidean norm of x_j. In the case $m = 1$, Φ is simply called radial. Obviously, if Φ is multiradial, so is its Fourier transform, which only depends on ϕ. Thus it is appropriate to use the notation

$$\mathcal{F}_{n_1,\dots,n_m}(\phi)(r_1, \dots, r_m) := F_{n_1,\dots,n_m}(\Phi)(\xi_1, \dots, \xi_m),$$

where $r_1 = |\xi_1|, \dots, r_m = |\xi_m|$, for the Fourier transform of a multiradial function Φ on $\mathbf{R}^{n_1+\cdots+n_m}$.

There exists an obvious identification between functions ϕ on $[0, \infty)^m$ and multi-even functions (functions that are even with respect to each of their variables) on $\mathbf{R}^m$ given by

$$\phi_{ext}(t_1, \dots, t_m) = \phi(|t_1|, \dots, |t_m|)\,.$$

Clearly, the restriction of ϕ_{ext} on $[0, \infty)^m$ is ϕ. We introduce the notation

$$\widehat{\phi} := F_{1,\dots,1}(\phi_{ext})\,.$$

Throughout this paper we denote the multi-even extension ϕ_{ext} of ϕ also by ϕ, and then $\widehat{\phi}$ provides a shorter notation for $F_{1,\dots,1}(\phi)$, which also coincides with $\mathcal{F}_{1,\dots,1}(\phi)$ on $[0, \infty)^m$.

In the recent work of Grafakos and Teschl [6] an explicit formula for the Fourier transform of a radial function $\Phi(x) = \phi(|x|)$ is given in terms of the one-dimensional Fourier transform of ϕ or the two-dimensional Fourier transform of $(t, s) \mapsto \phi(|(t, s)|)$. In this work we extend this formula to multiradial functions. We obtain relatively straightforward formulas that relate the Fourier transform on $\mathbf{R}^{m(k+2)}$ with that on $\mathbf{R}^{mk}$ but also new more complicated ones that relate the Fourier transform on $\mathbf{R}^{m(k+1)}$ with that on $\mathbf{R}^{mk}$; the latter formulas are valid only in the case of compactly supported Fourier transforms, i.e., band-limited multiradial signals.

We have the following results:

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Theorem 1.1. *Let $m \geq 1$ and $k_i \in \mathbf{Z}^+$ for $i = 1, \ldots, m$. Suppose that Φ is related to ϕ via* (1.1) *and that ϕ satisfies*

$$\int_{[0,\infty)^m} \prod_{j=1}^{m} (1 + r_i)^{2k_j+1} |\phi(r_1, \ldots, r_m)| dr < \infty.$$

Then the following identities are valid:

$$\begin{aligned}
&\mathcal{F}_{2k_1+1,\ldots,2k_m+1}(\phi)(r_1, \ldots, r_m)\\
&= \frac{1}{(2\pi)^{k_1+\cdots+k_m}} \sum_{\ell_m=1}^{k_m} \frac{(-1)^{\ell_m}(2k_m - \ell_m - 1)!}{2^{k_m-\ell_m}(k - \ell_m)!(\ell_m - 1)!} \frac{1}{r_m^{2k_m-\ell_m}}\\
&\quad \cdots \sum_{\ell_1=1}^{k_1} \frac{(-1)^{\ell_1}(2k_1 - \ell_1 - 1)!}{2^{k_1-\ell_1}(k_1 - \ell_1)!(\ell_1 - 1)!} \frac{1}{r_1^{2k_1-\ell_1}} \frac{\partial^{\ell_1+\cdots+\ell_m} \mathcal{F}_{1,\ldots,1}(\phi)}{\partial r_m^{\ell_m} \cdots \partial r_1^{\ell_1}}(r_1, \ldots, r_m)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{F}_{2k_1+2,\ldots,2k_m+2}(\phi)(r_1, \ldots, r_m)\\
&= \frac{1}{(2\pi)^{k_1+\cdots+k_m}} \sum_{\ell_m=1}^{k_m} \frac{(-1)^{\ell_m}(2k_m - \ell_m - 1)!}{2^{k_m-\ell_m}(k_m - \ell_m)!(\ell_m - 1)!} \frac{1}{r_m^{2k_m-\ell_m}}\\
&\quad \cdots \sum_{\ell_1=1}^{k_1} \frac{(-1)^{\ell_1}(2k_1 - \ell_1 - 1)!}{2^{k_1-\ell_1}(k_1 - \ell_1)!(\ell_1 - 1)!} \frac{1}{r_1^{2k_1-\ell_1}} \frac{\partial^{\ell_1+\cdots+\ell_m} \mathcal{F}_{2,\ldots,2}(\phi)}{\partial r_m^{\ell_m} \cdots \partial r_1^{\ell_1}}(r_1, \ldots, r_m).
\end{aligned}$$

Remark 1.2. We prove the identity

$$\mathcal{F}_{k_1+2,\ldots,k_m+2}(\phi)(r_1, \ldots, r_m) = \frac{(-1)^m}{(2\pi)^m r_1 \cdots r_m} \frac{\partial^m \mathcal{F}_{k_1,\ldots,k_m}(\phi)}{\partial r_m \cdots \partial r_1}(r_1, \ldots, r_m)$$

for every $k_i \in \mathbf{Z} \cup \{0\}$ and this can be iterated to give the claimed identities in Theorem 1.1.

Remark 1.3. The integrability assumption on ϕ allows us to consider the function Φ given by (1.1), and defined on $\mathbf{R}^n$ for any n satisfying $1 \leq n \leq 2(k_1 + \cdots + k_m + m)$. Then $\Phi \in L^1(\mathbf{R}^n)$. Using the fact the Fourier transform is a unitary operator on $L^2(\mathbf{R}^{n_1+\cdots+n_m})$ and by density, L^1-integrability of Φ in the above theorem can be replaced by L^2-integrability. About the associated recursion in Theorem 1.1 for the case of Schwartz functions, we refer the reader to [7, 10, 11] for related results. One could consider analogous recursion formulas for multiradial distributions; this has been studied in the linear case in [12, 14, 15].

Remark 1.4. We have given formulas for the Fourier transform of $\phi(|x_1|, \ldots, |x_m|)$ when either all x_i lie in odd-dimensional spaces or all x_i lie in even-dimensional spaces in terms of the Fourier transform on ϕ on $\mathbf{R}^m$ or $\mathbf{R}^{2m}$, respectively. Analogous formulas work for the Fourier transform of functions $\phi(|x_1|, \ldots, |x_m|)$ where $x_i \in \mathbf{R}^{n_i}$ in terms of the Fourier transform of $\phi(t_1, \ldots, t_m)$, where $t_i \in \mathbf{R}$ when n_i is odd and $t_i \in \mathbf{R}^2$ when n_i is even.

Theorem 1.5. *(a) Let ϕ be an even function on a real line whose Fourier transform $\widehat{\phi}$ is supported in the interval $[-A, A]$. Suppose that Φ is related to ϕ via* (1.1) *and that for some $k \in \mathbf{Z} \cup \{0\}$ we have*

$$\int_{[0,\infty)} (1 + r)^{2k+1} |\phi(r)| dr < \infty.$$

If $k = 0$, then the following identity is valid:

$$\mathcal{F}_2(\phi)(r) = 2 \int_r^A (\widehat{\phi})'(w) \frac{dw}{\sqrt{w^2 - r^2}} \chi_{[0,A]}(r). \tag{1.2}$$

When $k \geq 1$ we have

$$\mathcal{F}_{2k+1}(\phi)(r) = \frac{1}{(2\pi)^k} \sum_{\ell=1}^{k} \frac{(-1)^\ell(2k - \ell - 1)!}{2^{k-\ell}(k - \ell)!(\ell - 1)!} \frac{1}{r^{2k-\ell}} \frac{d^\ell \widehat{\phi}}{dw^\ell}(r) \chi_{(0,A)}(r)$$

and

$$\mathcal{F}_{2k+2}(\phi)(r) = \frac{2}{(2\pi)^k}\sum_{\ell=1}^{k}\frac{(-1)^\ell(2k-\ell-1)!}{2^{k-\ell}(k-\ell)!(\ell-1)!}\bigg(\int_r^A \frac{1}{w^{2k-\ell}}\frac{d^{\ell+1}\widehat{\phi}}{dw^{\ell+1}}(w)\frac{dw}{\sqrt{w^2-r^2}}\bigg)\chi_{(0,A)}(r)\,. \tag{1.3}$$

(b) Let $m \geq 2$ and let ϕ be a function defined on $\mathbf{R}^m$ which is even with respect to any variable. Suppose that the Fourier transform $\widehat{\phi}$ of ϕ is supported in $[-A, A]^m$. Let Φ be related to ϕ via (1.1) and suppose that for some $k_j \in \mathbf{Z}\cup\{0\}$ we have

$$\int_{[0,\infty)^m}\prod_{j=1}^{m}(1+r_j)^{2k_j+1}|\phi(r_1,\dots,r_m)|dr < \infty.$$

When all $k_j = 0$, then we have

$$\begin{aligned}&\mathcal{F}_{2,\dots,2}(\phi)(r_1,\dots,r_m)\\ &= 2^m\int_{r_m}^A\cdots\int_{r_1}^A \frac{\partial^m\widehat{\phi}}{\partial w_m\cdots\partial w_1}(w_1,\dots,w_m)\frac{dw_1}{\sqrt{w_1^2-r_1^2}}\cdots\frac{dw_m}{\sqrt{w_m^2-r_m^2}}\,\chi_{(0,A)^m}(r_1,\dots,r_m).\end{aligned} \tag{1.4}$$

If all $k_j \geq 1$ we have

$$\begin{aligned}&\mathcal{F}_{2k_1+1+\cdots+2k_m+1}(\phi)(r_1,\dots,r_m)\\ &= \frac{1}{(2\pi)^{k_1+\cdots+k_m}}\sum_{\ell_1=1}^{k_1}\frac{(-1)^{\ell_1}(2k_1-\ell_1-1)!}{2^{k_1-\ell_1}(k_1-\ell_1)!(\ell_1-1)!}\cdots\sum_{\ell_m=1}^{k_m}\frac{(-1)^{\ell_m}(2k_m-\ell_m-1)!}{2^{k_m-\ell_m}(k_m-\ell_m)!(\ell_m-1)!}\\ &\qquad \frac{1}{r_1^{2k_1-\ell_1}\cdots r_m^{2k_m-\ell_m}}\frac{\partial^{\ell_1+\cdots+\ell_m}\widehat{\phi}}{\partial r_1^{\ell_1}\cdots\partial r_m^{\ell_m}}(r_1,\dots,r_m)\chi_{(0,A)^m}(r_1,\dots,r_m)\end{aligned}$$

and

$$\begin{aligned}&\mathcal{F}_{2k_1+2,\dots,2k_m+2}(\phi)(r_1,\dots,r_m)\\ &= \frac{2^m}{(2\pi)^{k_1+\cdots+k_m}}\sum_{\ell_1=1}^{k_1}\frac{(-1)^{\ell_1}(2k_1-\ell_1-1)!}{2^{k_1-\ell_1}(k_1-\ell_1)!(\ell_1-1)!}\cdots\sum_{\ell_m=1}^{k_m}\frac{(-1)^{\ell_m}(2k_m-\ell_m-1)!}{2^{k_m-\ell_m}(k_m-\ell_m)!(\ell_m-1)!}\\ &\qquad \bigg(\int_{[r_1,A]}\cdots\int_{[r_m,A]}\frac{1}{w_1^{2k_1-\ell_1}\cdots w_m^{2k_m-\ell_m}}\frac{\partial^{\ell_1+\cdots+\ell_m+m}\widehat{\phi}}{\partial w_1^{\ell_1+1}\cdots\partial w_m^{\ell_m+1}}(w_1,\dots,w_m)\\ &\qquad\qquad\cdots\frac{dw_1}{\sqrt{w_1^2-r_1^2}}\frac{dw_m}{\sqrt{w_m^2-r_m^2}}\bigg)\chi_{(0,A)^m}(r_1,\dots,r_m)\,.\end{aligned}$$

Remark 1.6. We conclude the following: Under the hypotheses of the preceding theorem (part (b)), if $\mathcal{F}_{1,\dots,1}(\phi)$ has compact support, then so does $\mathcal{F}_{2,\dots,2}(\phi)$. More generally, by combining these two theorems, we also deduce that for every integers $k_1,\dots,k_m$ then $\mathcal{F}_{k_1,\dots,k_m}(\phi)$ has compact support too. This property can also be obtained as a consequence of the finite speed of propagation of the Euclidean Laplace operator $\Delta_{\mathbf{R}^n} = \otimes_{j=1}^m\Delta_{\mathbf{R}^{k_i}}$, see [1, Lemma 3.1]. Moreover, in the radial case this property can also be rephrased as follows: a Fourier band-limited function is also a Hankel band-limited function, for the "J_0" Hankel transform and refer the reader to [2, 8] for more details. The work of Rawn [8] also provided an inspiration for identity (1.2).

Remark 1.7. For Φ related to ϕ via (1.1), under the hypotheses of the preceding theorem (part (b)), we have an exact formula for its Fourier transform, only in terms of the Fourier transform of the function ϕ on $\mathbf{R}^1\times\cdots\times\mathbf{R}^1$.

We will also give some examples in the last section and describe an application to the framework of bilinear Marcinkiewicz-type Fourier multipliers. More precisely, we show that the transformation consisting to replace a bi-even bilinear kernel K on $\mathbf{R}$ by a bilinear kernel $\widetilde{K}$ on $\mathbf{R}^n$ with $\widetilde{K}(y,z) = (|y||z|)^{-n+1}K(|y|,|z|)$ preserves the Marcinkiewicz conditions (see Subsection 3.1 for details).

2. Proofs

Proof of Theorem 1.1. For simplicity of exposition, we only consider the case where $k_1 = \cdots = k_m = n$. The general case only presents notational differences. Throughout the proof we denote by J_ν the Bessel function of order ν and by $\widetilde{J}_\nu(t) = t^{-\nu} J_\nu(t)$.

Using polar coordinates, the Fourier transform of an integrable radial function Φ on $\mathbf{R}^{mn}$ is given by

$$\begin{aligned}
&F_{n,\dots,n}(\Phi)(\xi_1, \xi_2, \dots, \xi_m)\\
&= \int_0^\infty \cdots \int_0^\infty \phi(s_1, \dots, s_m) \int_{(S^{n-1})^m} e^{-2\pi i s \xi \cdot \theta}\, d\theta s_1 \cdots s_m ds_1 \cdots ds_m\\
&= (2\pi)^m \int_0^\infty \cdots \int_0^\infty \phi(s_1, \dots, s_m) J_{\frac{n}{2}-1}(2\pi s_1 |\xi_1|) \left(\frac{s_1}{|\xi_1|}\right)^{\frac{n}{2}-1} s_1 ds_1\\
&\qquad \cdots J_{\frac{n}{2}-1}(2\pi s_m |\xi_m|) \left(\frac{s_m}{|\xi_m|}\right)^{\frac{n}{2}-1} s_m ds_m\\
&= (2\pi)^{\frac{mn}{2}} \int_{[0,\infty]^m} \phi(s_1, \dots, s_m) \widetilde{J}_{\frac{n}{2}-1}(2\pi s_1 r_1) s_1^n \frac{ds_1}{s_1} \cdots \widetilde{J}_{\frac{n}{2}-1}(2\pi s_m r_m) s_m^n \frac{ds_m}{s_m}\\
&:= \mathcal{F}_{n,\dots,n}(\phi)(r_1, \dots, r_m),
\end{aligned}$$

where $|\xi_1| = r_1, \dots, |\xi_m| = r_m$.

A useful fact that will be used is that $\{-\frac{1}{2\pi}\frac{1}{r_i}\frac{\partial}{\partial r_i}\}_{i=1}^m$ commute for different values of i.

We differentiate $\mathcal{F}_{n,\dots,n}(\phi)(r_1, \dots, r_m)$ with respect with r_1. Using the identity

$$\frac{d}{dt}\widetilde{J}_\nu(t) = -t\widetilde{J}_{\nu+1}(t),$$

which holds for all $t > 0$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial r_1}\mathcal{F}_{n,\dots,n}(\phi)(r_1, \dots, r_m) = -(2\pi)^{\frac{mn}{2}+2} r_1 \int_0^\infty \cdots \int_0^\infty \phi(s_1, \dots, s_m)&\\
\widetilde{J}_{\frac{n+2}{2}-1}(2\pi s_1 r_1) s_1^{n+2-1} ds_1 \cdots \widetilde{J}_{\frac{n}{2}-1}(2\pi s_m r_m) s_m^{n-1} ds_m&.
\end{aligned}$$

Differentiating with respect to the remaining variables $r_2, \dots, r_m$ we obtain

$$\begin{aligned}
&\frac{\partial^m}{\partial r_m \cdots \partial r_1}(\mathcal{F}_{n,\cdots,n}(\phi))(r_1, \dots, r_m)\\
&= (-1)^m (2\pi)^{2m} (2\pi)^{\frac{mn}{2}} r_1 \cdots r_m \int_0^\infty \int_0^\infty \phi(s_1, \dots, s_m)\\
&\qquad \widetilde{J}_{\frac{n+2}{2}-1}(2\pi s_1 r_1) s_1^{n+2-1} ds_1 \cdots \widetilde{J}_{\frac{n+2}{2}-1}(2\pi s_m r_m) s_m^{n+2-1} ds_m\\
&= (-1)^m (2\pi)^m r_1 \cdots r_m \mathcal{F}_{n+2,\dots,n+2}(\phi)(r_1, \dots, r_m)
\end{aligned}$$

or

$$\begin{aligned}
\mathcal{F}_{n+2,\dots,n+2}(\phi)(r_1, \dots, r_m) &= (-1)^m \frac{1}{(2\pi)^m r_1 \cdots r_m} \frac{\partial^m \mathcal{F}_{n,\cdots,n}(\phi)}{\partial r_m \cdots \partial r_1}(r_1, \dots, r_m)\\
&= \Big(-\frac{1}{2\pi}\frac{1}{r_m}\frac{\partial}{\partial r_m}\Big) \cdots \Big(-\frac{1}{2\pi}\frac{1}{r_1}\frac{\partial}{\partial r_1}\Big)\mathcal{F}_{n,\dots,n}(\phi)(r_1, \dots, r_m). \qquad (2.1)
\end{aligned}$$

It is easy to check the interchanging differentiation and integration in the preceding calculations is permissible because of the hypothesis on the integrability of Φ which translates to a condition about the integrability of $\phi(s_1, \dots, s_m)(s_1^2 + \cdots + s_m^2)^{n-1}$ for all $n \le 2(mk + m)$.

For $k \in (\mathbf{Z}^+)^m$, using (2.1) by induction on n, starting with $n=1$, we obtain

$$\begin{aligned}
&\mathcal{F}_{2k_1+1,\dots,2k_m+1}(\phi)(r_1,\dots,r_m)\\
&\quad= \Big(-\frac{1}{2\pi}\frac{1}{r_m}\frac{\partial}{\partial r_m}\Big)^{k_m}\cdots\Big(-\frac{1}{2\pi}\frac{1}{r_1}\frac{\partial}{\partial r_1}\Big)^{k_1}(\mathcal{F}_{1,\dots,1}(\phi))(r_1,\dots,r_m)\\
&\quad= \Big(-\frac{1}{2\pi}\frac{1}{r_m}\frac{\partial}{\partial r_m}\Big)^{k_m}\cdots\Big(-\frac{1}{2\pi}\frac{1}{r_2}\frac{\partial}{\partial r_2}\Big)^{k_2}\\
&\qquad\quad \sum_{\ell_1=1}^{k_1}\frac{(-1)^{\ell_1}(2k_1-\ell_1-1)!}{2^{k_1-\ell_1}(k_1-\ell_1)!(\ell_1-1)!}\frac{1}{r_1^{2k_1-\ell_1}}\frac{\partial^{\ell_1}\mathcal{F}_{1,\dots,1}(\phi)}{\partial r_1^{\ell_1}}(r_1,\dots,r_m)\\
&\quad= \frac{1}{(2\pi)^{k_1+\cdots+k_m}}\sum_{\ell_m=1}^{k_m}\frac{(-1)^{\ell_m}(2k_m-\ell_m-1)!}{2^{k_m-\ell_m}(k_m-\ell_m)!(\ell_m-1)!}\frac{1}{r_m^{2k_m-\ell_m}}\\
&\qquad\quad \cdots\sum_{\ell_1=1}^{k_1}\frac{(-1)^{\ell_1}(2k_1-\ell_1-1)!}{2^{k_1-\ell_1}(k_1-\ell_1)!(\ell_1-1)!}\frac{1}{r_1^{2k_1-\ell_1}}\frac{\partial^{\ell_1+\cdots+\ell_m}\mathcal{F}_{1,\dots,1}(\phi)}{\partial r_m^{\ell_m}\cdots\partial r_1^{\ell_1}}(r_1,\dots,r_m)
\end{aligned}$$

and likewise we obtain

$$\begin{aligned}
&\mathcal{F}_{2k_1+2,\dots,2k_1+2}(\phi)(r_1,\dots,r_m)\\
&\quad= \frac{1}{(2\pi)^{k_1+\cdots+k_m}}\sum_{\ell_m=1}^{k_1}\frac{(-1)^{\ell_m}(2k_m-\ell_m-1)!}{2^{k_m-\ell_m}(k_m-\ell_m)!(\ell_m-1)!}\frac{1}{r_m^{2k-\ell_m}}\\
&\qquad\quad \cdots\sum_{\ell_1=1}^{k_1}\frac{(-1)^{\ell_1}(2k_1-\ell_1-1)!}{2^{k_1-\ell_1}(k_1-\ell_1)!(\ell_1-1)!}\frac{1}{r_1^{2k_1-\ell_1}}\frac{\partial^{\ell_1+\cdots+\ell_m}\mathcal{F}_{2,\dots,2}(\phi)}{\partial r_m^{\ell_m}\cdots\partial r_1^{\ell_1}}(r_1,\dots,r_m).
\end{aligned}$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.5. We prove this theorem with $A=\pi$. If this case is proved, then we can take $\phi_0(t)=\frac{\pi}{A}\phi(\frac{\pi}{A}t)$ and by a change of variables we obtain (1.2) and (1.3) in Theorem 1.5.

Step 1. It is a well known fact (see [4]) that

$$F_2(\Phi)(\xi)=2\pi\int_0^\infty \phi(s)J_0(2\pi s|\xi|)s\,ds=\mathcal{F}_2(\phi)(r)\,, \tag{2.2}$$

where $J_0(t)=\frac{1}{\pi}\int_{-1}^1 e^{ist}\frac{ds}{\sqrt{1-s^2}}$ is the Bessel function of order zero.

In this step, we want to prove that given ϕ even function on the real line, there exists one and only one function f on a real line such that

$$\phi(x)=\int_0^\pi f(u)J_0(2\pi ux)u\,du. \tag{2.3}$$

First, we look for necessary conditions on f, to be a solution of (2.3). So momentarily assume that such an f exists, by applying a change of variables and Fubini's theorem, we obtain

$$\begin{aligned}
\int_0^\pi f(u)J_0(2\pi ux)u\,du &= \frac{1}{\pi}\int_0^\pi f(u)u\int_{-1}^1 e^{i2\pi uxs}\frac{ds}{\sqrt{1-s^2}}\,du\\
&= \frac{1}{\pi}\int_0^\pi f(u)u\int_{-u}^u e^{i2\pi wx}\frac{dw}{\sqrt{u^2-w^2}}du\\
&= \int_{-\pi}^\pi e^{2\pi iwx}\left\{\frac{1}{\pi}\int_{|w|}^\pi f(u)u\frac{du}{\sqrt{u^2-w^2}}\right\}dw.
\end{aligned}\tag{2.4}$$

Thus, we rewrite (2.3) as

$$\phi(x) = \int_{-\pi}^{\pi} e^{2\pi i w x} \left\{ \frac{1}{\pi} \int_{|w|}^{\pi} f(u) \frac{u du}{\sqrt{u^2 - w^2}} \right\} dw. \tag{2.5}$$

On the other hand, recalling that $\widehat{\phi}$ is supported in $[-\pi, \pi]$, we have $\phi(x) = \int_{-\pi}^{\pi} \widehat{\phi}(w) e^{2\pi i w x} dw$ and thus by identifying with (2.4), it comes

$$\widehat{\phi}(w) = \frac{1}{\pi} \int_{|w|}^{\pi} f(u) \frac{u du}{\sqrt{u^2 - w^2}}. \tag{2.6}$$

Since ϕ is even, so is $\widehat{\phi}$, thus it is sufficient to deal with the case $w > 0$.

Integrating both sides of (2.6) with respect to $\frac{w dw}{\sqrt{w^2 - y^2}}$ we obtain

$$h(y) := \int_y^{\pi} \widehat{\phi}(w) \frac{w dw}{\sqrt{w^2 - y^2}} = \frac{1}{\pi} \int_y^{\pi} \int_w^{\pi} f(u) \frac{u du}{\sqrt{u^2 - w^2}} \frac{w dw}{\sqrt{w^2 - y^2}}. \tag{2.7}$$

But an easy change of variables shows that $\int_y^u \frac{w dw}{\sqrt{w^2 - y^2}\sqrt{u^2 - w^2}} = \frac{\pi}{2}$. Then applying Fubini's theorem, we deduce

$$h(y) = \frac{1}{\pi} \int_y^{\pi} f(u) u \int_y^u \frac{w dw}{\sqrt{u^2 - w^2}\sqrt{w^2 - y^2}} du = \frac{1}{2} \int_y^{\pi} f(u) u du. \tag{2.8}$$

Combining (2.7) with (2.8), we get

$$\int_y^{\pi} f(u) u du = 2 \int_y^{\pi} \widehat{\phi}(w) \frac{w dw}{\sqrt{w^2 - y^2}}. \tag{2.9}$$

We integrate by parts in (2.9), recalling the support of $\widehat{\phi}$, and differentiating with respect to y we obtain

$$\begin{aligned} -f(y) y &= 2 \frac{d}{dy} \left(\sqrt{\pi^2 - y^2} \widehat{\phi}(\pi) - \int_y^{\pi} \sqrt{w^2 - y^2} (\widehat{\phi})'(w) dw \right) \\ &= -2 \int_y^{\pi} \frac{y}{\sqrt{w^2 - y^2}} (\widehat{\phi})'(w) dw \end{aligned}$$

thus

$$f(y) = 2 \int_y^{\pi} (\widehat{\phi})'(w) \frac{dw}{\sqrt{w^2 - y^2}}. \tag{2.10}$$

Once this calculation is done, it is quite easy to check that the function f given in (2.10) satisfies (2.3) by reversing the preceding steps. Moreover, the previous computations yield that this solution of (2.3) is the only one.

Step 2. For functions ϕ such that $\int_0^{\infty} |\phi(s)| s\, ds < \infty$ we define an operator

$$U(\phi)(r) = \int_0^{\infty} \phi(s) J_0(2\pi s r)\, s ds.$$

We want to prove the identity

$$U^2(\phi)(t) = \frac{1}{2\pi} \phi(t). \tag{2.11}$$

To prove (2.11), it is enough to show that for all $t > 0$ we have

$$\int_0^{\infty} \int_0^{\infty} \phi(s) J_0(2\pi s r) s ds J_0(2\pi r t) r dr = \frac{1}{2\pi} \phi(t). \tag{2.12}$$

We start with the identity (see [13] page 406)

$$t \int_0^{\infty} J_1(2\pi t r) J_0(2\pi s r) dr = \begin{cases} 1 & s < t, \\ 0 & s > t. \end{cases} \tag{2.13}$$

Multiplying (2.13) by $\phi(s)s$ and integrating from 0 to ∞, we obtain

$$\int_0^\infty \phi(s)st\int_0^\infty J_1(2\pi tr)J_0(2\pi sr)drds = \int_0^t \phi(s)sds. \tag{2.14}$$

Using that $\frac{d}{du}(u^\nu J_\nu(u)) = u^\nu J_{\nu-1}(u)$, and differentiating both sides of (2.14) with respect to t, we get

$$\int_0^\infty \phi(s)s\int_0^\infty 2\pi tr J_0(2\pi tr)J_0(2\pi sr)drds = \phi(t)t.$$

This proves (2.12) and hence (2.11).

Step 3. In view of the result of Step 1, there exists a function f such that

$$\begin{aligned}\mathcal{F}_2(\phi)(r) &= 2\pi\int_0^\infty \phi(s)J_0(2\pi sr)sds\\ &= 2\pi\int_0^\infty\int_0^\infty f(u)\chi_{[0,\pi]}(u)J_0(2\pi su)udu J_0(2\pi sr)sds\\ &= f(r)\chi_{[0,\pi]}(r)\\ &= 2\int_r^\pi (\widehat{\phi})'(w)\frac{dw}{\sqrt{w^2-r^2}}\chi_{[0,\pi]}(r). \end{aligned}\tag{2.15}$$

which proves (1.2).

Combining (2.15) with the result of Theorem 1.1. when $m = 1$, we obtain

$$\begin{aligned}\mathcal{F}_4(\phi)(r) &= -\frac{1}{2\pi}\frac{1}{r}\frac{d}{dr}(\mathcal{F}_2(\phi))(r)\\ &= -2\frac{1}{2\pi}\frac{1}{r}\frac{d}{dr}\left(-\int_r^\pi \frac{d}{dw}\Big(\frac{(\widehat{\phi})'(w)}{w}\Big)\sqrt{w^2-r^2}dw\right)\chi_{(0,\pi)}(r)\\ &= \frac{2}{2\pi}\left(\int_r^\pi \frac{d}{dw}\Big(\frac{(\widehat{\phi})'(w)}{w}\Big)\frac{dw}{\sqrt{w^2-r^2}}\right)\chi_{(0,\pi)}(r). \end{aligned}\tag{2.16}$$

Differentiating (2.16) $k-1$ times, we obtain (1.3) with $A = \pi$. Due to symmetry of ϕ, the other formula in Theorem 1.5 is directly deduced from the first equation in Theorem 1.1.

We now proceed to part (b). For simplicity we look at the case where $m = 2$ and $A = \pi$.

Step 1. For Φ on $\mathbf{R}^4$ and $\xi \in \mathbf{R}^2$, $\eta \in \mathbf{R}^2$

$$\begin{aligned}F_{2,2}(\Phi)(\xi,\eta) &= \int_0^\infty\int_0^\infty \phi(s_1,s_2)\int_{S^1}\int_{S^1} e^{-2\pi s_1\eta\cdot\theta_1}e^{-2\pi s_2\xi\cdot\theta_2}d\theta_1 d\theta_2 s_1 s_2 ds_1 ds_2\\ &= (2\pi)^2\int_0^\infty\int_0^\infty \phi(s_1,s_2)J_0(2\pi s_1|\xi|)s_1ds_1J_0(2\pi s_2|\eta|)s_2ds_2\\ &:= \mathcal{F}_{2,2}(\phi)(r_1,r_2),\end{aligned}$$

where $\Phi(\xi,\eta) = \phi(|\xi|,|\eta|)$, $J_0(t) = \frac{1}{\pi}\int_{-1}^1 e^{ist}\frac{ds}{\sqrt{1-s^2}}$ and $|\xi| = r_1$, $|\eta| = r_2$.

We proceed as for the part (a). So we first aim to show that there exists a unique function f on $[0,\pi]^2$ such that

$$\phi(x_1,x_2) = \int_0^\pi\int_0^\pi f(u_1,u_2)J_0(2\pi u_1x_1)J_0(2\pi u_2x_2)u_1u_2du_1du_2. \tag{2.17}$$

Assume momentarily that such a function exists. For a function h we have

$$\begin{aligned}\int_0^\pi h(u)J_0(2\pi ux)udu &= \frac{1}{\pi}\int_0^\pi h(u)u\int_{-1}^1 e^{2\pi iuxs}\frac{ds}{\sqrt{1-s^2}}du\\ &= \frac{1}{\pi}\int_0^\pi h(u)u\int_{-u}^u e^{2\pi iwx}\frac{dw}{\sqrt{u^2-w^2}}du\\ &= \int_{-\pi}^\pi e^{2\pi iwx}\left\{\frac{1}{\pi}\int_{|w|}^\pi h(u)u\frac{du}{\sqrt{u^2-w^2}}\right\}dw. \end{aligned}\tag{2.18}$$

Thus, we rewrite (2.17) as

$$\phi(x_1,x_2) = \frac{1}{\pi^2}\int_{-\pi}^\pi\int_{-\pi}^\pi e^{2\pi iw_1x_1}e^{2\pi iw_2x_2}\left\{\int_{|w_2|}^\pi\int_{|w_1|}^\pi f(u_1,u_2)\frac{u_1du_1}{\sqrt{u_1^2-w_1^2}}\frac{u_2du_2}{\sqrt{u_2^2-w_2^2}}\right\}dw_1dw_2.$$

Recalling the support of $\widehat{\phi}$, we have $\phi(x_1,x_2) = \int_{-\pi}^\pi\int_{-\pi}^\pi \widehat{\phi}(w_1,w_2)e^{2\pi i(w_1x_1+w_2x_2)}dw_1dw_2$. Thus the function f on $\mathbf{R}^2$ would satisfy:

$$\widehat{\phi}(w_1,w_2) = \frac{1}{\pi^2}\int_{|w_2|}^\pi\int_{|w_1|}^\pi f(u_1,u_2)\frac{u_1du_1}{\sqrt{u_1^2-w_1^2}}\frac{u_2du_2}{\sqrt{u_2^2-w_2^2}}.\tag{2.19}$$

Since ϕ is even, it is sufficient to consider the case $w_1,w_2>0$.
Then integrating both sides of (2.19) with respect to $\frac{w_2dw_2}{\sqrt{w_2^2-y_2^2}}\frac{w_1dw_1}{\sqrt{w_1^2-y_1^2}}$ we obtain

$$\begin{aligned}h(y_1,y_2) :&= \int_{y_1}^\pi\int_{y_2}^\pi \widehat{\phi}(w_1,w_2)\frac{w_2dw_2}{\sqrt{w_2^2-y_2^2}}\frac{w_1dw_1}{\sqrt{w_1^2-y_1^2}}\\ &= \frac{1}{\pi^2}\int_{y_1}^\pi\int_{y_2}^\pi\int_{w_2}^\pi\int_{w_1}^\pi f(u_1,u_2)\frac{u_1du_1}{\sqrt{u_1^2-w_1^2}}\frac{u_2du_2}{\sqrt{u_2^2-w_2^2}}\frac{w_2dw_2}{\sqrt{w_2^2-y_2^2}}\frac{w_1dw_1}{\sqrt{w_1^2-y_1^2}}.\end{aligned}\tag{2.20}$$

Note that $\int_y^u \frac{wdw}{\sqrt{w^2-y^2}\sqrt{u^2-w^2}} = \frac{\pi}{2}$. Applying Fubini's theorem three times, we get

$$\begin{aligned}h(y_1,y_2) &= \frac{1}{\pi^2}\int_{y_1}^\pi\int_{y_2}^\pi\left\{\int_{w_1}^\pi f(u_1,u_2)\frac{u_1du_1}{\sqrt{u_1^2-w_1^2}}\right\}\int_{y_2}^{u_2}\frac{w_2dw_2}{\sqrt{w_2^2-y_2^2}\sqrt{u_2^2-w_2^2}}u_2du_2\frac{w_1dw_1}{\sqrt{w_1^2-y_1^2}}\\ &= \frac{1}{2\pi}\int_{y_1}^\pi\int_{y_2}^\pi\left\{\int_{w_1}^\pi f(u_1,u_2)\frac{u_1du_1}{\sqrt{u_1^2-w_1^2}}\right\}u_2du_2\frac{w_1dw_1}{\sqrt{w_1^2-y_1^2}}\\ &= \frac{1}{2\pi}\int_{y_2}^\pi\left\{\int_{y_1}^\pi\int_{w_1}^\pi f(u_1,u_2)\frac{u_1du_1}{\sqrt{u_1^2-w_1^2}}\frac{w_1dw_1}{\sqrt{w_1^2-y_1^2}}\right\}u_2du_2\\ &= \frac{1}{4}\int_{y_2}^\pi\int_{y_1}^\pi f(u_1,u_2)u_1du_1u_2du_2.\end{aligned}\tag{2.21}$$

Using (2.19) and and (2.21), we deduce

$$\int_{y_2}^\pi\int_{y_1}^\pi f(u_1,u_2)u_1du_1u_2du_2 = 4\int_{y_1}^\pi\int_{y_2}^\pi \widehat{\phi}(w_1,w_2)\frac{w_2dw_2}{\sqrt{w_2^2-y_2^2}}\frac{w_1dw_1}{\sqrt{w_1^2-y_1^2}}.\tag{2.22}$$

We can recover f from this equation. Differentiating (2.22) with respect with y_1 and y_2, we obtain

$$f(y_1,y_2)y_1y_2$$
$$= 4\frac{\partial^2}{\partial y_2 \partial y_1}\left(\int_{y_1}^{\pi}\left\{\int_{y_2}^{\pi}\widehat{\phi}(w_1,w_2)\frac{w_2dw_2}{\sqrt{w_2^2-y_2^2}}\right\}\frac{w_1dw_1}{\sqrt{w_1^2-y_1^2}}\right)$$
$$= 4\frac{\partial^2}{\partial y_2 \partial y_1}$$
$$\left(\sqrt{\pi^2-y_1^2}\int_{y_2}^{\pi}\widehat{\phi}(\pi,w_2)\frac{w_2dw_2}{\sqrt{w_2^2-y_2^2}} - \int_{y_1}^{\pi}\sqrt{w_1^2-y_1^2}\left\{\int_{y_2}^{\pi}\frac{\partial\widehat{\phi}}{\partial w_1}(w_1,w_2)\frac{w_2dw_2}{\sqrt{w_2^2-y_2^2}}\right\}dw_1\right).$$

Recalling the support of $\widehat{\phi}$, we get

$$f(y_1,y_2)y_1y_2$$
$$= 4\frac{\partial^2}{\partial y_2 \partial y_1}$$
$$\left(-\int_{y_1}^{\pi}\sqrt{w_1^2-y_1^2}\left\{\sqrt{\pi^2-y_2^2}\frac{\partial\widehat{\phi}}{\partial w_1}(\pi,w_2)-\int_{y_2}^{\pi}\sqrt{w_2^2-y_2^2}\frac{\partial^2\widehat{\phi}}{\partial w_2\partial w_1}(w_1,w_2)dw_2\right\}dw_1\right)$$
$$= 4\frac{\partial^2}{\partial y_2 \partial y_1}\left(\int_{y_1}^{\pi}\sqrt{w_1^2-y_1^2}\int_{y_2}^{\pi}\sqrt{w_2^2-y_2^2}\frac{\partial^2\widehat{\phi}}{\partial w_2\partial w_1}(w_1,w_2)dw_2dw_1\right)$$
$$= 4\int_{y_1}^{\pi}\frac{y_1}{\sqrt{w_1^2-y_1^2}}\int_{y_2}^{\pi}\frac{y_2}{\sqrt{w_2^2-y_2^2}}\frac{\partial^2\widehat{\phi}}{\partial w_2\partial w_1}(w_1,w_2)dw_2dw_1$$

or

$$f(y_1,y_2) = 4\int_{y_1}^{\pi}\int_{y_2}^{\pi}\frac{\partial^2\widehat{\phi}}{\partial w_2\partial w_1}(w_1,w_2)\frac{dw_2}{\sqrt{w_2^2-y_2^2}}\frac{dw_1}{\sqrt{w_1^2-y_1^2}}.$$

We notice that this function f we have constructed in this way satisfies (2.17) by reversing the preceding steps and is the unique solution.

Step 2. For functions ϕ on $\mathbf{R}^2$ such that $\int_0^\infty\int_0^\infty|\phi(s_1,s_2)|s_1s_2ds<\infty$, we define an operator U by setting

$$U(\phi)(r_1,r_2) = \int_0^\infty\int_0^\infty \phi(s_1,s_2)J_0(2\pi s_1r_1)s_1ds_1J_0(2\pi s_2r_2)s_2ds_2.$$

We want to prove the following identity

$$U^2(\phi)(t_1,t_2) = \frac{1}{(2\pi)^2}\phi(t_1,t_2). \tag{2.23}$$

It is enough to show

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \phi(s_1,s_2)J_0(2\pi s_1r_1)s_1ds_1J_0(2\pi s_2r_2)s_2ds_2J_0(2\pi r_1t_1)r_1dr_1J_0(2\pi r_2t_2)r_2dr_2$$
$$= \frac{1}{(2\pi)^2}\phi(t_1,t_2).$$

We make use of the fact below that can be found in [13] page 406:

$$t_2t_1\int_0^\infty\int_0^\infty J_1(2\pi t_1r_1)J_0(2\pi s_1r_1)dr_1J_1(2\pi t_2r_2)J_0(2\pi s_2r_2)dr_2 = \begin{cases}1 & \text{if } s_1<t_1 \text{ and } s_2<t_2.\\ 0 & \text{otherwise.}\end{cases}$$

Multiplying the preceding identity by $\phi(s_1, s_2)s_1s_2$, integrating both sides in s_1 and s_2, we obtain

$$\begin{aligned}&\int_0^\infty\int_0^\infty \phi(s_1,s_2)s_1s_2t_2t_1\int_0^\infty\int_0^\infty J_1(2\pi t_1r_1)J_0(2\pi s_1r_1)dr_1J_1(2\pi t_2r_2)J_0(2\pi s_2r_2)dr_2ds_1ds_2\\ &\qquad=\int_0^{t_2}\int_0^{t_1}\phi(s_1,s_2)s_1s_2ds_1ds_2. \end{aligned}\tag{2.24}$$

By applying $\frac{d}{du}(u^\nu J_\nu(u)) = u^\nu J_{\nu-1}(u)$, and differentiating both sides of (2.24) with respect to t_1 and t_2, we obtain

$$\begin{aligned}&\int_0^\infty\int_0^\infty \phi(s_1,s_2)s_1s_2\int_0^\infty\int_0^\infty(2\pi r_1t_1)J_0(2\pi t_1r_1)J_0(2\pi s_1r_1)dr_1\\ &\qquad(2\pi r_2t_2)J_0(2\pi t_2r_2)J_0(2\pi s_2r_2)dr_2ds_1ds_2\\ &=\phi(t_1,t_2)t_1t_2.\end{aligned}$$

which proves (2.23).

Step 3. Using the results of the Step 1 and 2, there exists a function f on $\mathbf{R}^2$ such that

$$\begin{aligned}\mathcal{F}_{2,2}(\phi)(r_1,r_2)&=(2\pi)^2\int_0^\infty\int_0^\infty\phi(s_1,s_2)J_0(2\pi s_1r_1)s_1ds_1J_0(2\pi s_2r_2)s_2ds_2\\ &=(2\pi)^2\int_0^\infty\int_0^\infty\int_0^\pi\int_0^\pi f(u_1,u_2)J_0(2\pi u_1sx_1)J_0(2\pi u_2s_2)u_1u_2du_1du_2\\ &\qquad J_0(2\pi s_1r_1)s_1ds_1J_0(2\pi s_2r_2)s_2ds_2\\ &=f(r_1,r_2)\chi_{[-\pi,\pi]\times[-\pi,\pi]}(r_1,r_2)\\ &=4\int_{r_2}^\pi\int_{r_1}^\pi\frac{\partial^2\widehat{\phi}}{\partial w_2\partial w_1}(w_1,w_2)\frac{dw_1}{\sqrt{w_1^2-r_1^2}}\frac{dw_2}{\sqrt{w_2^2-r_2^2}}\chi_{[0,\pi]\times[0,\pi]}(r_1,r_2)\end{aligned}$$

which proves (1.4) when $m = 2$.

Applying (2.1) with $m = 2, n = 2$, we obtain

$$\begin{aligned}\mathcal{F}_{4,4}(\phi)(r_1,r_2)&=\Big(-\frac{1}{2\pi}\frac{1}{r_2}\Big)\Big(-\frac{1}{2\pi}\frac{1}{r_1}\Big)\frac{\partial^2}{\partial r_2\partial r_1}\left\{\mathcal{F}_{2,2}(\phi)(r_1,r_2)\right\}\\ &=4\Big(-\frac{1}{2\pi}\frac{1}{r_2}\Big)\Big(-\frac{1}{2\pi}\frac{1}{r_1}\Big)\frac{\partial^2}{\partial r_2\partial r_1}\left\{\int_{r_2}^\pi\int_{r_1}^\pi\frac{\partial^2\widehat{\phi}}{\partial w_2\partial w_1}\frac{dw_1}{\sqrt{w_1^2-r_1^2}}\frac{dw_2}{\sqrt{w_2^2-r_2^2}}\right\}\\ &=4\Big(-\frac{1}{2\pi}\frac{1}{r_2}\Big)\Big(-\frac{1}{2\pi}\frac{1}{r_1}\Big)\frac{\partial^2}{\partial r_2\partial r_1}\\ &\qquad\left\{\int_{r_2}^\pi\int_{r_1}^\pi\frac{\partial}{\partial w_2}\left(\frac{1}{w_2}\frac{\partial}{\partial w_1}\left(\frac{1}{w_1}\frac{\partial^2\widehat{\phi}}{\partial w_2\partial w_1}\right)\right)\sqrt{w_1^2-r_1^2}dw_1\sqrt{w_2^2-r_2^2}dw_2\right\}\\ &=4\frac{1}{(2\pi)^2}\int_{r_2}^\pi\int_{r_1}^\pi\frac{\partial}{\partial w_2}\left(\frac{1}{w_2}\frac{\partial}{\partial w_1}\left(\frac{1}{w_1}\frac{\partial^2\widehat{\phi}}{\partial w_2\partial w_1}\right)\right)\frac{dw_1}{\sqrt{w_1^2-r_1^2}}\frac{dw_2}{\sqrt{w_2^2-r_2^2}},\end{aligned}$$

where $(r_1, r_2) \in (0, \pi) \times (0, \pi)$.

Iterating this procedure, we complete the proof when $m = 2$. The case of general m presents only notational differences and can be easily deduced by induction. □

3. Applications and Examples

3.1. **Applications to bilinear Marcinkiewicz operators.** Let us first recall the setting of bilinear Fourier multipliers. On $\mathbf{R}^n$, a bilinear operator T acting from $\mathcal{S}(\mathbf{R}^n)\times\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}'(R^n)$ is a bilinear Fourier multiplier if it commutes with the simultaneous translations. Equivalently, there exist a bilinear kernel $K \in \mathcal{S}'(\mathbf{R}^{2n})$ and a bilinear symbol $m \in \mathcal{S}'(\mathbf{R}^{2n})$ such that for every smooth functions $f, g, h \in \mathcal{S}(\mathbf{R}^n)$ we have the two following representations:

$$\begin{aligned}\langle T(f,g), h\rangle &= \int_{\mathbf{R}^{3n}} K(y,z) f(x-y) g(x-z) h(x)\, dx\, dy\, dz \\ &= \int_{\mathbf{R}^{2n}} m(\xi,\eta) \widehat{f}(\xi) \widehat{g}(\eta) \widehat{h}(\xi+\eta)\, d\xi\, d\eta.\end{aligned}$$

The kernel K and the symbol m are related by the Fourier transform $K = \widehat{m}$. We denote by T_K the bilinear operator associated to the kernel K.

Then consider a bi-even bilinear kernel K on $\mathbf{R}^2$ and exponents $p_1, p_2 \geq 1$ such that the bilinear operator T_K is bounded from $L^{p_1}(\mathbf{R}) \times L^{p_2}(\mathbf{R})$ into $L^p(\mathbf{R})$, where p is given by the Hölder scaling $p^{-1} = p_1^{-1} + p_2^{-1}$. Now for $n \geq 2$, we may consider the bilinear kernel defined on $\mathbf{R}^n$ by

$$\widetilde{K}(y,z) = (|z||y|)^{-(n-1)} K(|y|, |z|),$$

where the factor $(|z||y|)^{-(n-1)}$ is implicitly dictated by the Hölder scaling. A natural question arises: which assumptions allow us to transport the $(L^{p_1}(\mathbf{R}) \times L^{p_2}(\mathbf{R}) \to L^p(\mathbf{R}))$-boundedness of T_K to a $(L^{p_1}(\mathbf{R}^n) \times L^{p_2}(\mathbf{R}^n) \to L^p(\mathbf{R}^n))$-boundedness of $T_{\widetilde{K}}$?
That would correspond to the bilinear version of results in [3], where such a question is studied in the linear setting.

To answer such a question, it could be first interesting to see how this transformation $K \to \widetilde{K}$ acts on different classes of bilinear operators which are known to be bounded, such as bilinear Calderón-Zygmund operators, and bilinear multiplier operators whose symbols satisfy the Hörmander or the Marcinkiewicz condition. It is obvious that the Calderón-Zygmund conditions on the kernel are not preserved by the transformation $K \to \widetilde{K}$.

Using the previous results, we can begin to give a positive answer in the setting of bilinear Marcinkiewicz operators. Let us first recall that a bilinear Fourier multiplier T_K is called of Marcinkiewicz type if its bilinear symbol m satisfies the following regularity condition:

$$\sup_{\xi,\eta} |\xi|^{|\alpha|} |\eta|^{|\beta|} \left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)\right| \leq C_{\alpha,\beta}, \tag{3.1}$$

for every multi-indices α, β.

Then we have the following:

Proposition 3.1. *If T_K is a bilinear Fourier multiplier on $\mathbf{R}$ of Marcinkiewicz type then for every odd dimension $n \geq 3$, the bilinear operator $T_{\widetilde{K}}$ is also a bilinear Fourier multiplier of Marcinkiewicz type on $\mathbf{R}^n$.*

Proof. Let $\tilde{m}$ the bilinear symbol associated to $\widetilde{K}$. So

$$\widetilde{m}(\xi,\eta) = \widehat{\widetilde{K}}(\xi,\eta) = \mathcal{F}_{n,n}((r_1 r_2)^{-(n-1)} K)(|\xi|, |\eta|),$$

and we have (since K is assumed to be multi-even)

$$\mathcal{F}_{1,1}((r_1 r_2)^{-(n-1)} K)(r_1, r_2) = M^n(r_1, r_2),$$

where M^n is the $(n-1)^{th}$-primitive of the symbol m (on each coordinate) given by

$$M^n(r_1,r_2) = \left(\int_0^{r_1} \int_0^{t_{n-1}} \cdots \int_0^{t_2}\right) \left(\int_0^{r_2} \int_0^{s_{n-1}} \cdots \int_0^{s_2}\right) m(t_1, s_1)\, dt_1 ... dt_{n-1} ds_1 ... ds_{n-1}.$$

Applying Theorem 1.1, it comes that since m satisfies the regularity property (3.1) in $\mathbf{R}$, then $\widetilde{m}$ satisfies the same in $\mathbf{R}^n$.
Indeed, Theorem 1.1 yields that $\widetilde{m}$ is a sum of terms of the form

$$\frac{1}{|\xi|^{2k-\ell_1} |\eta|^{2k-\ell_2}} \frac{\partial^{\ell_1+\ell_2}}{\partial r_1^{\ell_1} \partial r_2^{\ell_2}} M^n(|\xi|, |\eta|).$$

However the regularity on m implies the following estimates on M^n

$$\sup_{r_1,r_2} r_1^{\alpha-(n-1)} r_2^{\beta-(n-1)} \left|\partial_{r_1}^{\alpha}\partial_{r_2}^{\beta} M^n(r_1,r_2)\right| \lesssim C_{\alpha,\beta},$$

hence we deduce that $\widetilde{m}$ is of Marcinkiewicz type on $\mathbf{R}^n$. $\square$

We refer the reader to [5] by the second author and Kalton, where they studied the boundedness of bilinear Marcinkiewicz-type Fourier multipliers. More precisely in [5, Theorem 7.3], a criterion is found to be almost equivalent to the boundedness from $L^{p_1} \times L^{p_2}$ into L^p and it is surprising to see that this criterion does not depend on p_1, p_2, p. It could be interesting to develop this approach and study if this criterion is preserved by our transformation $K \to \widetilde{K}$.

We also refer the reader to [3] where a similar result was proved in the linear case via a similar idea. A minor difference is that the following companion recurrence formula in [4] on page 425

$$\frac{d}{dt}(t^{\nu} J_{\nu}(t)) = t^{\nu} J_{\nu-1}(t)$$

was used in the proof of [3, Theorem 1.8], which results in a recursion formula which is decreasing in the dimension.

3.2. **Examples.** The following facts are known; see for instance Appendix C in [9]. For $a, b > 0$ and $x, \xi \in \mathbf{R}^1$, the Fourier transform of

$$f(x) = \begin{cases} \dfrac{\cos(b\sqrt{a^2-x^2}\,)}{\sqrt{a^2-x^2}} & \text{if } |x| < a \\ 0 & \text{if } |x| > a \end{cases}$$

is the function $\xi \mapsto \pi J_0(a\sqrt{b^2+4\pi^2\xi^2})$ and the Fourier transform of

$$g(x) = \begin{cases} \dfrac{\cosh(b\sqrt{a^2-x^2}\,)}{\sqrt{a^2-x^2}} & \text{if } |x| < a \\ 0 & \text{if } |x| > a \end{cases}$$

is

$$G(\xi) = \begin{cases} \pi J_0(a\sqrt{4\pi^2\xi^2-b^2}) & \text{if } 2\pi|\xi| > b \\ \pi J_0(ai\sqrt{b^2-4\pi^2\xi^2}) & \text{if } 2\pi|\xi| < b. \end{cases} \tag{3.2}$$

Another useful formula is that if $h(x) = \dfrac{\sin(b\sqrt{a^2+x^2}\,)}{\sqrt{a^2+x^2}}$, then

$$\widehat{h}(\xi) = \begin{cases} \pi J_0(a\sqrt{b^2-4\pi^2\xi^2}) & \text{if } |2\pi\xi| < b \\ 0 & \text{if } |2\pi\xi| > b. \end{cases} \tag{3.3}$$

We have the following examples:

Example 1. On $\mathbf{R}^{2n}$ consider the function

$$\Phi(x,y) = \frac{\cos(\sqrt{4\pi^2-|x|^2}\sqrt{4\pi^2+|y|^2})}{\sqrt{4\pi^2-|x|^2}} \chi_{(0,2\pi)}(|x|)\chi_{(0,2\pi)}(|y|)$$

Clearly $\Phi(x,y) = \phi(|x|,|y|)$ for some function ϕ on $\mathbf{R}^2$. Obviously, $\Phi \in L^1(\mathbf{R}^{2n})$ for all $n \geq 1$.

First, we fix $y \in \mathbf{R}^1$, and then using the first formula of the preceding facts we calculate that the Fourier transform of Φ associated with the first variable on $\mathbf{R}^1$ is

$$\widehat{\Phi_y}(\xi, y) = \pi J_0(2\pi\sqrt{4\pi^2+y^2+4\pi^2\xi^2})\chi_{(0,2\pi)}(|y|).$$

Second, applying the inverse version of the first formula and the convolution theorem of Fourier transforms, we get that the Fourier transform of Φ on $\mathbf{R}^2$ is

$$F_{1,1}(\Phi)(\xi,\eta) = \left\{ \frac{\cos(4\pi^2\sqrt{1+|\xi|^2}\sqrt{1-|\cdot|^2}\,)}{\sqrt{1-|\cdot|^2}} \chi_{(0,1)}(|\cdot|) \right\} * \left\{ \frac{1}{|\cdot|}\sin(4\pi^2|\cdot|) \right\}(\eta),$$

where the convolution is in the one-dimensional dotted variable. By an easy change of variables, we rewrite the preceding formula as

$$\mathcal{F}_{1,1}(\phi)(r_1,r_2)=\left\{\frac{\cos(4\pi^2\sqrt{1+r_1^2}\sqrt{1-|\cdot|^2}\,)}{\sqrt{1-|\cdot|^2}}\chi_{(0,1)}(|\cdot|)\right\}*\left\{\frac{1}{|\cdot|}\sin(4\pi^2|\cdot|)\right\}(r_2),$$

where $|\xi|=r_1$ and $|\eta|=r_2$.

Note that

$$(-\frac{1}{2\pi r_2}\frac{\partial}{\partial r_2})(-\frac{1}{2\pi r_1}\frac{\partial}{\partial r_1})\left[\frac{\cos(4\pi^2\sqrt{1+r_1^2}\sqrt{1-r_2^2}\,)}{\sqrt{1-r_2^2}}\right]=\frac{4\pi^2\cos(4\pi^2\sqrt{1+r_1^2}\sqrt{1-r_2^2}\,)}{\sqrt{1-r_2^2}}.$$

Finally using (2.1) with $m=2,n=1$, after an algebraic manipulation and in view of the identity $\frac{d}{dr}(f*g)(r)=(\frac{df}{dr}*g)(r)$, we obtain that on $\mathbf{R}^{3\times 3}$ we have

$$F_{3,3}(\Phi)(\xi,\eta)=\left\{\frac{4\pi^2\cos(4\pi^2\sqrt{1+|\xi|^2}\sqrt{1-|\cdot|^2}\,)}{\sqrt{1-|\cdot|^2}}\chi_{(0,1)}(|\cdot|)\operatorname{sgn}(\cdot)\right\}*\left\{\frac{1}{|\cdot|}\sin(4\pi^2|\cdot|)\right\}(|\eta|),$$

where $\xi\in\mathbf{R}^3,\eta\in\mathbf{R}^3$ and the convolution is in the one-dimensional dotted variable.

Next we have an example in the case $n_1\neq n_2$.

Example 2. For $x\in\mathbf{R}^2$ and $y\in\mathbf{R}$ set

$$\Phi(x,y)=\begin{cases}\dfrac{\cosh(\sqrt{4\pi^2-|x|^2}\sqrt{4\pi^2-y^2})}{\sqrt{4\pi^2-|x|^2}} & \text{when } |x|<2\pi,\ |y|<2\pi,\\ 0 & \text{otherwise.}\end{cases}$$

Obviously, $\Phi\in L^1(\mathbf{R}^n)$ for all $n\geq 3$ and $\Phi(x,y)$ has the form $\phi(|x|,|y|)$ for some function ϕ on $\mathbf{R}^2$.

By the same argument as in Example 1, indeed making use of (3.2), (3.3) and the inverse version of (3.3) respectively, we obtain

$$\mathcal{F}_{2,1}(\phi)(r_1,r_2)=2\pi^2\Big(J_0\Big(4\pi^2\sqrt{r_1^2-1}\sqrt{1-|\cdot|^2}\Big)\chi_{(0,1)}(|\cdot|)\Big)*\Big(\frac{1}{|\cdot|}\sin(4\pi^2|\cdot|)\Big)(r_2).$$

Applying the identity $\frac{d}{dr}J_0(r)=-J_1(r)$, $\frac{d}{dr}J_1(r)=r^{-1}J_1(r)-J_2(r)$ from B.2 (1) in [4], it follows from a small modification of (2.1) that $F_{4,3}(\Phi)(\xi,\eta)$ is equal to

$$\left\{\left(\frac{4\pi^2J_1(4\pi^2\sqrt{|\xi|^2-1}\sqrt{1-|\cdot|^2}\,)}{\sqrt{|\xi|^2-1}\sqrt{1-|\cdot|^2}}-8\pi^4\sqrt{|\xi|^2-1}\,J_2(4\pi^2\sqrt{|\xi|^2-1}\sqrt{1-|\cdot|^2})\right)\chi_{(0,1)}(|\cdot|)\operatorname{sgn}(\cdot)\right\}$$
$$*\left\{\frac{1}{|\cdot|}\sin(4\pi^2|\cdot|)\right\}(|\eta|),$$

on $\mathbf{R}^{4\times 3}$ where $\xi\in\mathbf{R}^4,\eta\in\mathbf{R}^3$. Again the convolution is one-dimensional.

The following example shows how to obtain the two-dimensional Fourier transform of a radial function whose corresponding one-dimensional Fourier transform is compactly supported.

Example 3. For $t\in\mathbf{R}$, consider the even function

$$\phi(t)=\frac{\sin(2\pi\sqrt{1+t^2}\,)}{\sqrt{1+t^2}}$$

and define a square-integrable function on $\mathbf{R}^2$ by setting $\Phi(x)=\phi(|x|)$. Applying (3.3) we obtain

$$\widehat{\phi}(\tau)=\pi J_0\big(2\pi\sqrt{1-|\tau|^2}\,\big)\chi_{|\tau|<1}$$

for $\tau\in\mathbf{R}$. Then we apply (1.2) to deduce that for $r\in[0,1)$ we have

$$\mathcal{F}_2(\phi)(r)=2\pi\int_r^1\frac{d}{dt}J_0\big(2\pi\sqrt{1-t^2}\,\big)\frac{dt}{\sqrt{t^2-r^2}}=(2\pi)^2\int_r^1 J_1\big(2\pi\sqrt{1-t^2}\,\big)\frac{t}{\sqrt{1-t^2}}\frac{dt}{\sqrt{t^2-r^2}}\,,$$

where the last identity is due to the fact that $J_0' = J_{-1} = -J_1$. Setting $u = \sqrt{1-t^2}$ we rewrite the preceding integral as

$$\mathcal{F}_2(\phi)(r) = (2\pi)^2 \int_0^{\sqrt{1-r^2}} J_1\big(2\pi u\big) \frac{du}{\sqrt{1-r^2-u^2}} = -(2\pi)^2 \int_0^1 J_{-1}\big(2\pi\sqrt{1-r^2}\, t\big) \frac{dt}{\sqrt{1-t^2}}\,.$$

Using the identity B.3 in [4] (with $\mu = -1$, $\nu = -1/2$) the preceding expression is equal to

$$\Gamma(1/2)2^{-1/2} \frac{J_{-1/2}\big(2\pi\sqrt{1-r^2}\,\big)}{\big(2\pi\sqrt{1-r^2}\,\big)^{1/2}} = \frac{\cos\big(2\pi\sqrt{1-r^2}\,\big)}{2\pi\sqrt{1-r^2}}\,.$$

This provides a formula for the two-dimensional Fourier transform $\widehat{\Phi}$ of Φ as a function of $r = |\xi|$ when $r \in [0, 1)$. Notice that $\widehat{\Phi}(\xi)$ vanishes when $|\xi| \geq 1$.

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The identity is only stated for $\mu > -1/2$ but it is also valid for $\mu > -3/2$ by analytic continuation.